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Eigenstates of the higher powers of an annihilation operator for a time-dependent harmonic oscillator and their properties

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Abstract
In the framework of dynamical invariant theory, we introduce the annihilation operator $\hat{a}(t)$ and creation operator $\hat{a}^{\dagger}(t)$ for a time-dependent harmonic oscillator. The orthonormalized eigenstates of the operator $\hat{a}^{N}(t)(N \geqslant 3)$ are constructed, and their mathematical and quantum statistical properties are discussed in detail. Specifically, a particle moving in a Paul trap with periodically varying frequency is treated as an example to demonstrate the dynamical squeezing effects for the eigenstates of the operator $\hat{a}^{3}(t)$.

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1. Introduction

The squeezing effect is one of the most typical non-classical properties in a quantized light field, and has an important application prospect in optical communication and gravitational-wave detections. Since the implement of the squeezed light experimentally in 1985 [1, 2], much attention has been paid to non-classical optical fields (see, e.g., [3, 4]). In order to study the non-classical effects of optical fields and their possible applications, various quantized optical states with non-classical effects should be beforehand prepared.

It is well known that the eigenstates of the annihilation operator $\hat{a}$, i.e. the coherent state, of quantized optical fields, are typically classical quantum states. These states cannot show any non-classical effect, and their behaviors are most similar to the classic ones of optical fields. However, the usual even and odd coherent states (i.e. the eigenstate of $\hat{a}^2$) are obvious non-classical states [5]: there is no squeezing effect but photon anti-bunching effects are clearly shown in the odd coherent states, while there exist obvious squeezing effects but photons are bunching in the even coherent states. Later, much attention has been paid to investigating the quantum statistical properties of the orthogonal normalized eigenstates of the operators $\hat{a}^N$ [6–9] with $N \geqslant 3$. For example, the properties of quantum fluctuations on two quadrature components of phase amplitudes for three orthogonal normalized eigenstates of $\hat{a}^3$ are specifically discussed in [6]. The results show that these three states are all the non-classical light field states: the photons are obvious anti-bunching, although there is no squeezing effect.

The harmonic oscillator is a simple but typical idealized model in physics. It provides a suitable description of many practical problems, such as a quantized optical field, the vibrations of lattices, the motion of the particle in a Paul trap [10], etc. In particular, when a particle either oscillates in a dissipation environment or moves in a trap with changing parameters, its motion could be described by a time-dependent harmonic oscillator (TDHO). The dynamical invariant method, introduced first by Lewis and Riesenfeld [11], is one of most suitable ones to solve the quantum dynamical problem for a TDHO. In the framework of this theory, some new concepts including time-dependent coherent states [12, 13] and dynamical squeezing [14] are defined by introducing the so-called time-dependent bosonic creation operator $\hat{a}^\dagger(t)$ and annihilation one $\hat{a}(t)$. It is emphasized that the time-dependent coherent states $|z, t\rangle$ [12, 13], i.e. the instantaneous eigenstates of the operator $\hat{a}(t)$, reveal completely different quantum statistical behaviors from the usual time-independent coherent states. Similarly, the instantaneous eigenstates of the operator $\hat{a}^2(t)$ have also different features (e.g. the dynamical squeezing) from the usual even and odd coherent states of the time-independent optical fields [15, 16].

In this paper, we will investigate the properties of the orthogonal normalized eigenstates of $\hat{a}^N(t)$. First, a dynamical invariant for a generic TDHO is constructed by making use of two special solutions to its classical dynamics, and the desirable time-dependent annihilation operator $\hat{a}(t)$ is accordingly introduced for such a TDHO. Next, we construct the orthogonal normalized eigenstates of the operator $\hat{a}^N(t)$, and investigate their mathematical and quantum statistical properties. As an example, we finally discuss the non-classical properties of the eigenstates of the operator $\hat{a}^3(t)$ in detail for the harmonic oscillator with a periodically varying frequency. It is shown that, obvious squeezing effects are indicated in these states, which are different from the results that no squeezing effect is shown in the eigenstates of the time-independent bosonic operator $\hat{a}^3$ [6, 9].

2. A dynamical invariant for a time-dependent harmonic oscillator

We consider a general TDHO with the following Hamiltonian ($\hbar = 1$ throughout this paper):

$$\hat{H}(t) = \frac{1}{2}[Z(t)\hat{p}^2 + X(t)\hat{q}^2]. \tag{1}$$

Its dynamical evolution is determined by the time-dependent Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial(t)}|\varphi(t)\rangle = \hat{H}(t)|\varphi(t)\rangle. \tag{2}$$

A dynamical invariant operator $\hat{I}(t)$, introduced first by the Lewis and Riesenfeld [11], should satisfy the conditions

$$\hat{I}^\dagger(t) = \hat{I}(t), \qquad \frac{\partial\hat{I}(t)}{\partial t} + \mathrm{i}[\hat{H}(t), \hat{I}(t)] = 0. \tag{3}$$

The exact solution to the TDHO (1) can be constructed in terms of the instantaneous eigenstates of the invariant operator $\hat{I}(t)$. Obviously, dynamical invariant determined by equation (3) is not sole, even if for a common Hamiltonian. Indeed, by making use of two arbitrary linearly independent solutions $x_1(t)$ and $x_2(t)$ to the classical equation of motion

$$\ddot{q}_{cl} - \frac{\dot{Z}(t)}{Z(t)}\dot{q}_{cl} + [X(t)Z(t)]q_{cl} = 0 \tag{4}$$

for the TDHO (1), we can easily construct a desirable invariant [13, 17]

$$\hat{I}(t) = \tfrac{1}{2}\{e^{\alpha(t)}(\hat{p} - \beta(t)\hat{q})^2 - \gamma^2 e^{-\alpha(t)}\hat{q}^2\} = \hat{W}(t)\hat{I}_0\hat{W}^\dagger(t) \tag{5}$$

with $\alpha(t) = \ln[x_1(t)x_2(t)]$, $\beta(t) = \dot{\alpha}(t)/[2Z(t)]$, $\hat{I}_0 = (\hat{p}^2 - \gamma^2\hat{q}^2)/2$, $\hat{W}(t) = \exp\{(i/2)\beta(t)\hat{q}^2 - [i\alpha(t)/4][\hat{p}\hat{q} + \hat{q}\hat{p}]\}$ where $\gamma = [1/2Z(t)][x_1(t)\dot{x_2}(t) - x_2(t)\dot{x_1}(t)] =$ const. Certainly, two linearly independent conjugate solutions $x(t) = x_1(t) + ix_2(t)$ and $x^*(t)$ can also be selected. In this case, $\gamma = ik$ with k being a real number. As a consequence, one can introduce a pair of creation and annihilation operators

$$\hat{a}(t) = \frac{1}{\sqrt{2k}}\left\{k\, e^{\frac{-\alpha(t)}{2}}\hat{q} + i\, e^{\frac{\alpha(t)}{2}}[\hat{p} - \beta(t)\hat{q}]\right\} = \hat{W}(t)\hat{a}_0\hat{W}^\dagger(t), \tag{6}$$

$$\hat{a}^\dagger(t) = \frac{1}{\sqrt{2k}}\left\{k\, e^{\frac{-\alpha(t)}{2}}\hat{q} - i\, e^{\frac{\alpha(t)}{2}}[\hat{p} - \beta(t)\hat{q}]\right\} = \hat{W}(t)\hat{a}_0^\dagger\hat{W}^\dagger(t), \tag{7}$$

to rewrite the above dynamical invariant as

$$\hat{I}(t) = \tfrac{1}{2}\{e^{\alpha(t)}[\hat{p} - \beta(t)\hat{q}]^2 + k^2 e^{-\alpha(t)}\hat{q}^2\} = \left[\hat{a}^\dagger(t)\hat{a}(t) + \tfrac{1}{2}\right]k. \tag{8}$$

Above, $\hat{a}_0 = (1/\sqrt{2k})(k\hat{q} + i\hat{p})$ and $\hat{a}_0^\dagger = (1/\sqrt{2k})(k\hat{q} - i\hat{p})$.

Solving the instantaneous eigenvalue problem of the dynamical invariant operator (8): $\hat{I}(t)|n,t\rangle = \lambda_n|n,t\rangle$, we have $\lambda_n = (n + 1/2)k$ $(n = 0, 1, 2, \ldots)$ and $|n,t\rangle = \hat{W}(t)|n\rangle$. Here, $|n\rangle = [\sqrt{k}/(2^n n!\sqrt{\pi})]^{1/2}\exp(-\xi^2/2)H_n(\xi)$ with $\xi = \sqrt{k}q$. Finally, a general solution to the Schrödinger equation (2) for the TDHO (1) reads [11, 17]

$$|\varphi(t)\rangle = \sum_{n=0}^{\infty} C_n\, e^{i\theta_n(t)}|n,t\rangle, \tag{9}$$

with $C_n = \langle n, 0|\varphi(0)\rangle$ and

$$\theta_n(t) = \int_0^t \langle n, t'|i\frac{\partial}{\partial t'} - \hat{H}(t')|n, t'\rangle\, dt' = -\left(n + \frac{1}{2}\right)k\int_0^t \frac{z(t')}{|x(t')|^2}\, dt', \tag{10}$$

being the so-called Lewis–Riesenfeld phase.

3. Instantaneous eigenstates of the operator $\hat{a}^N(t)$

With a complete set spanned by the eignstates of the time-dependent Hermitian operator $\hat{I}(t)$, we introduce the following series of quantum states:

$$|\psi_j(z,t)\rangle_N = c_{N,j}\sum_{n=0}^{\infty}\frac{z^{Nn+j}\, e^{i\theta_{Nn+j}(t)}}{\sqrt{(Nn+j)!}}|Nn+j, t\rangle, \qquad j = 0, 1, 2 \ldots, N-1, \quad z = |z|e^{i\phi} \tag{11}$$

where

$$\theta_n(t) = \theta_{n-m}(t) + 2m\theta_0(t), \qquad n > m = 0, 1, 2 \ldots, \tag{12}$$

and $c_{N,j}$ is the normalization coefficient. Note that

$${}_N\langle\psi_j(z,t)|\psi_j(z,t)\rangle_N = c_{N,j}^2\sum_{n=o}^{\infty}\frac{y^{Nn+j}}{(Nn+j)!} = c_{N,j}^2 F_{N,j}(y) = 1, \tag{13}$$

with $y = |z|^2$ and $F_{N,j}(y) = \sum_{n=o}^{\infty} y^{Nn+j}/(Nn+j)!$, we have

$$F_{N,j}(y) = \frac{d^{(N-j)}}{dy^{(N-j)}}F_{N,0}(y), \qquad \sum_{j=0}^{N-1}F_{N,j}(y) = \sum_{n=0}^{\infty}\frac{y^n}{n!} = e^y. \tag{14}$$

After solving the differential equation

$$\sum_{j=0}^{N-1} \frac{\mathrm{d}^{(N-1-j)}}{\mathrm{d}y^{(N-1-j)}} F_{N,0}(y) = \mathrm{e}^{y}, \tag{15}$$

to determine the coefficient $F_{N,0}(y)$, all the other coefficients $F_{N,j}(y)$ can be obtained. Obviously, for $N = 1$, the quantum state constructed above reduces to the time-dependent coherent states [12, 13]: $|z,t\rangle = |\psi_0(z,t)\rangle_1 = c_{1,0}\sum_{n=o}^{\infty}\{z^n \exp[\mathrm{i}\theta_n(t)]\}/(\sqrt{n!})|n,t\rangle$, which are the instantaneous eigenstates of the operator $\hat{a}(t)$ in equation (6). Also, for $N = 2$, the states $|\psi_0(z,t)\rangle_2$ and $|\psi_1(z,t)\rangle_2$ are just the time-dependent even and odd coherent states [16, 18], respectively. Generally, it is easy to check that, the quantum states $|\psi_j(z,t)\rangle_N$ introduced above are the eigenstates of the operators $\hat{a}^N(t)$, corresponding to the eigenvalues $\{z\exp[\mathrm{i}2\theta_0(t)]\}^N$: $\hat{a}^N(t)|\psi_j(z,t)\rangle_N = \{z\exp[\mathrm{i}2\theta_0(t)]\}^N|\psi_j(z,t)\rangle_N$.

Clearly, for a fixed value of N the eigenstates of $\hat{a}^N(t)$ for different values of j are mutually orthogonal:

$$_N\langle\psi_j(z,t)|\psi_k(z,t)\rangle_N = 0, \qquad j,k = 0,1,2\ldots,N-1, \qquad j \neq k. \tag{16}$$

But, they are related by the actions of operator $\hat{a}(t)$, e.g.,

$$\begin{aligned}
&\hat{a}(t)|\psi_0(z,t)\rangle_N = z\,\mathrm{e}^{\mathrm{i}2\theta_0(t)} F_{N,0}{}^{-\frac{1}{2}} F_{N,N-1}{}^{\frac{1}{2}}|\psi_{N-1}(z,t)\rangle_N,\\
&\hat{a}^2(t)|\psi_0(z,t)\rangle_N = (z\,\mathrm{e}^{\mathrm{i}2\theta_0(t)})^2 F_{N,0}{}^{-\frac{1}{2}} F_{N,N-2}{}^{\frac{1}{2}}|\psi_{N-2}(z,t)\rangle_N,\\
&\ldots\\
&\hat{a}^j(t)|\psi_0(z,t)\rangle_N = (z\,\mathrm{e}^{\mathrm{i}2\theta_0(t)})^j F_{N,0}{}^{-\frac{1}{2}} F_{N,N-j}{}^{\frac{1}{2}}|\psi_{N-j}(z,t)\rangle_N,\\
&\ldots\\
&\hat{a}^N(t)|\psi_0(z,t)\rangle_N = (z\,\mathrm{e}^{\mathrm{i}2\theta_0(t)})^N \hat{F}_{N,0}^{-\frac{1}{2}}\hat{F}_{N,0}^{\frac{1}{2}}|\psi_0(z,t)\rangle_N = (z\,\mathrm{e}^{\mathrm{i}2\theta_0(t)})^N|\psi_0(z,t)\rangle_N.
\end{aligned} \tag{17}$$

It is well known that the Glauber coherent states $|z\rangle$ themselves are not orthogonal when z takes different values in the complex parameters plane, i.e. $|\langle z|z'\rangle|^2 = \mathrm{e}^{-|z-z'|^2} \neq 0$. Similarly, each of the time-dependent eigenstates $|\psi_j(z,t)\rangle_N$ of the operators $\hat{a}^N(t)$ is also not orthogonal for different values of z:

$$\begin{aligned}
_N\langle\psi_j(z,t)|\psi_j(z',t)\rangle_N &= [F_{N,j}(|z|^2)F_{N,j}(|z'|^2)]^{\frac{1}{2}}\sum_{n=0}^{\infty}\frac{(z^*z')^{Nn+j}}{(Nn+j)!}\\
&= [F_{N,j}(|z|^2)F_{N,j}(|z'|^2)]^{\frac{1}{2}}F_{N,j}(z^*z') \neq 0.
\end{aligned} \tag{18}$$

However, all the instantaneous eigenstates of the operator $\hat{a}^N(t)$ form a complete set. Indeed, one can easily prove that there exists the following completeness relation:

$$\begin{aligned}
\int\frac{\mathrm{d}^2z}{\pi}\left(\sum_{j=0}^{N-1}|\psi_j(z,t)\rangle_{NN}\langle\psi_j(z,t)|\right) &= \int\frac{\mathrm{d}^2z}{\pi}\,\mathrm{e}^{-y}\sum_{n=0}^{\infty}\frac{y^{Nn+j}}{\sqrt{(Nn+j)!}}|z,t\rangle\langle z,t|z,t\rangle\langle z,t|,\\
&= \int\frac{\mathrm{d}^2z}{\pi}\left\{\int\frac{\mathrm{d}^2z}{\pi}|z,t\rangle\langle z,t|\right\}|z,t\rangle\langle z,t| = 1.
\end{aligned} \tag{19}$$

Therefore, the orthonormalized eigenstates of the operator $\hat{a}^N(t)$ can span a complete Hilbert space.

4. Quantum statistical properties of the eigenstates of the operator $\hat{a}^N(t)$: a TDHO with periodically varying frequency

To examine the quantum statistical properties of the above quantum states, we introduce two basic quadrature operators [18]

$$\begin{aligned}\hat{X}_1(t) &= \tfrac{1}{2}\,\mathrm{e}^{\alpha(t)/2}[\hat{a}(t)+\hat{a}^\dagger(t)],\\ \hat{X}_2(t) &= \tfrac{1}{2}\,\mathrm{e}^{\alpha(t)/2}\{[(k^{-1}\beta(t)-\mathrm{i}\,\mathrm{e}^{-\alpha(t)})\hat{a}(t)]+[(k^{-1}\beta(t)+\mathrm{i}\,\mathrm{e}^{-\alpha(t)})\hat{a}^\dagger(t)]\}.\end{aligned} \tag{20}$$

It is not difficult to verify that they satisfy the commute relation $[\hat{X}_1(t),\hat{X}_2(t)]=\mathrm{i}/2$. For the arbitrary quantum state, the fluctuations of $\hat{X}_1(t)$ and $\hat{X}_2(t)$ can be expressed as

$$\langle\triangle X_1^2(t)\rangle = \tfrac{1}{4}\,\mathrm{e}^{\alpha(t)}\{\langle 2\hat{a}^\dagger(t)\hat{a}(t)+1+\hat{a}^{\dagger 2}(t)+\hat{a}^2(t)\rangle+\langle\hat{a}(t)+\hat{a}^\dagger(t)\rangle^2\} \tag{21}$$

and

$$\begin{aligned}\langle\triangle X_2^2(t)\rangle = \tfrac{1}{4}\{&\langle(k^{-2}\beta^2(t)\,\mathrm{e}^{\alpha(t)}+\mathrm{e}^{-\alpha(t)})[2\hat{a}^\dagger(t)\hat{a}(t)+1]+(k^{-1}\beta(t)\,\mathrm{e}^{\alpha(t)/2}+\mathrm{i}\,\mathrm{e}^{-\alpha(t)/2})^2\hat{a}^{\dagger 2}(t)\\ &+(k^{-1}\beta(t)\,\mathrm{e}^{\alpha(t)/2}-\mathrm{i}\,\mathrm{e}^{\alpha(t)/2})^2\hat{a}^2(t)\rangle+\langle(k^{-1}\beta(t)\,\mathrm{e}^{\alpha(t)/2}+\mathrm{i}\,\mathrm{e}^{-\alpha(t)/2})\hat{a}(t)\\ &+(k^{-1}\beta(t)\,\mathrm{e}^{\alpha(t)/2}-\mathrm{i}\,\mathrm{e}^{-\alpha(t)/2})\hat{a}^\dagger(t)\rangle^2\},\end{aligned} \tag{22}$$

respectively. If $\langle\triangle X_1^2(t)\rangle<1/4$ (or $\langle\triangle X_2^2(t)\rangle<1/4$), then the fluctuation on $X_1(t)$ (or $X_2(t)$) is squeezed [12]. With the help of the following relations

$$\begin{aligned}&{}_N\langle\psi_j(z,t)|\hat{a}^\dagger(t)|\psi_j(z,t)\rangle_N=\langle\psi_j(z,t)|\hat{a}(t)|\psi_j(z,t)\rangle=0, \qquad j=0,1,2\ldots N-1;\\ &{}_N\langle\psi_j(z,t)|\hat{a}^{\dagger 2}(t)+\hat{a}^2(t)|\psi_j(z,t)\rangle_N=\begin{cases}2y\cos[4\theta_0(t)+2\phi], & j=0,1\\ 0, & j\geqslant 2;\end{cases}\\ &{}_N\langle\psi_j(z,t)|\hat{a}^\dagger(t)\hat{a}(t)|\psi_j(z,t)\rangle_N=\begin{cases}yc_{N,0}^2c_{N,N-1}^2, & j=0\\ yc_{N,j}^2c_{N,j-1}^2, & j=1,2,\ldots,N-1,\end{cases}\end{aligned} \tag{23}$$

one can easily calculate the above quantum fluctuations for the state $|\psi_j(z,t)\rangle$:

$$\langle\triangle X_1^2(t)\rangle=\begin{cases}\frac{1}{4}\,\mathrm{e}^{\alpha(t)}\{2yc_{N,0}^2c_{N,N-1}^2+1+2y\cos[4\theta_0(t)+2\phi]\}, & j=0;\\ \frac{1}{4}\,\mathrm{e}^{\alpha(t)}\{2yc_{N,1}^2c_{N,0}^2+1+2y\cos[4\theta_0(t)+2\phi]\}, & j=1;\\ \frac{1}{4}\,\mathrm{e}^{\alpha(t)}(2yc_{N,j}^2c_{N,j-1}^2+1), & j\geqslant 2.\end{cases} \tag{24}$$

$$\langle\triangle X_2^2(t)\rangle=\begin{cases}\frac{1}{4}\{[k^{-2}\beta^2(t)\,\mathrm{e}^{\alpha(t)}+\mathrm{e}^{-\alpha(t)}](2yc_{N,0}^2c_{N,N-1}^2+1)+2[k^{-2}\beta^2(t)\,\mathrm{e}^{\alpha(t)}-\mathrm{e}^{-\alpha(t)}]y\\ \quad\times\cos[4\theta_0(t)+2\phi]+4k^{-1}\beta(t)y\sin[4\theta_0(t)+2\phi]\}, & j=0;\\ \frac{1}{4}\{[k^{-2}\beta^2(t)\,\mathrm{e}^{\alpha(t)}+\mathrm{e}^{-\alpha(t)}](2yc_{N,1}^2c_{N,0}^2+1)+2[k^{-2}\beta^2(t)\,\mathrm{e}^{\alpha(t)}-\mathrm{e}^{-\alpha(t)}]y\\ \quad\times\cos[4\theta_0(t)+2\phi]+4k^{-1}\beta(t)y\sin[4\theta_0(t)+2\phi]\}, & j=1;\\ \frac{1}{4}[k^{-2}\beta^2(t)\,\mathrm{e}^{\alpha(t)}+\mathrm{e}^{-\alpha(t)}](2yc_{N,j}^2c_{N,j-1}^2+1), & j\geqslant 2.\end{cases} \tag{25}$$

So, whether there is a squeezing effect for the eigenstates of the operator $\hat{a}^N(t)$ is a complex problem, which depends on the time-dependent parameters for the specific TDHO.

Without loss of generality, let us consider a typical TDHO with the classical equations of motion

$$\ddot{q}_{cl}+\omega^2(t)q_{cl}=0, \qquad \omega(t+T)=\omega(t), \tag{26}$$

where T is the period of the variable frequency. If the frequency takes the form $\omega^2(t)=a+2q\cos(2t)$, then equation (26) describes the classical motion of a particle in a Paul trap

[19]. If the condition $|q/(a-1)| < 1$ is satisfied, then the periodically varying frequency of the TDHO reduces to

$$\omega^2(t) = 1 + \frac{\xi(1-\eta^2)}{[1+\eta\cos(2t)]^2}, \tag{27}$$

where $\xi = (a-1)^3/[(a-1)^2 - q^2]$ and $\eta = -q/(a-1)$. With two specific complex solutions of the classical equation (26) of motion with frequency (27) [20]

$$x_1(t) = \sqrt{\frac{1+\eta\cos(2t)}{1+\eta}} \exp\left\{-\mathrm{i}\frac{\sqrt{1+\xi}}{2} \arcsin\left[\frac{\sqrt{1-\eta^2}\sin(2t)}{1+\eta\cos(2t)}\right]\right\}, \qquad x_2(t) = x_1^*(t), \tag{28}$$

the parameters in equation (5) can be specifically expressed as

$$\alpha(t) = \ln\left(\frac{1+\eta\cos(2t)}{1+\eta}\right), \qquad \beta(t) = \frac{-\eta\sin(2t)}{1+\eta\cos(2t)}, \qquad \gamma = \mathrm{i}k, \qquad k = \sqrt{\frac{(1-\eta)(1+\xi)}{(1+\eta)}},$$
$$\theta_n(t) = -\left(n+\frac{1}{2}\right)k\int_0^t \frac{(1+\eta)dt'}{1+\eta\cos(2t')} = \left(n+\frac{1}{2}\right)\sqrt{1+\xi}\arctan\left\{\frac{(\eta-1)}{\sqrt{1-\eta^2}}\tan(t)\right\}. \tag{29}$$

The quantum statistical properties of the instantaneous eigenstates of the operators $\hat{a}(t)$ and $\hat{a}^2(t)$ had been discussed in [10, 11]. Now, compared with the quantum statistical properties of the eigenstates of the time-independent operator $\hat{a}^3$ [6], we will investigate those properties of the eigenstates of the time-dependent operator $\hat{a}^3(t)$ for the TDHO with frequency (27). Specifically, three orthogonal eigenstates of the operator $\hat{a}^3(t)$ (with the same eigenvalue $\{z\exp[\mathrm{i}2\theta_0(t)]\}^3$) read

$$|\psi_j(z,t)\rangle_3 = c_{3,j}\sum_{n=0}^{\infty}\frac{z^{3n+j}\,\mathrm{e}^{\mathrm{i}\theta_{3n+j}(t)}}{\sqrt{(3n+j)!}}|3n+j,t\rangle, \qquad j=0,1,2. \tag{30}$$

Solving the second-order ordinary differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}F_{3,0}(y) + \frac{\mathrm{d}}{\mathrm{d}y}F_{3,0}(y) + F_{3,0}(y) = \mathrm{e}^y, \tag{31}$$

we get the normalized coefficient $c_{3,0} = F_{3,0}^{-1/2}$ and then $c_{3,1}$, $c_{3,2}$ by using equation (13):

$$\begin{aligned} c_{3,0} &= \left[\tfrac{1}{3}\,\mathrm{e}^y + \tfrac{2}{3}\,\mathrm{e}^{-y/2}\cos(\sqrt{3}y/2)\right]^{-1/2} \\ c_{3,1} &= \left[\tfrac{1}{3}\,\mathrm{e}^y - \tfrac{1}{3}\,\mathrm{e}^{-y/2}\cos(\sqrt{3}y/2) + \tfrac{1}{3}\,\mathrm{e}^{-y/2}\sin(\sqrt{3}y/2)\right]^{-1/2} \\ c_{3,2} &= \left[\tfrac{1}{3}\,\mathrm{e}^y - \tfrac{1}{3}\,\mathrm{e}^{-y/2}\cos(\sqrt{3}y/2) - \tfrac{1}{3}\,\mathrm{e}^{-y/2}\sin(\sqrt{3}y/2)\right]^{-1/2}. \end{aligned} \tag{32}$$

For these three eigenstates $|\psi_j(z,t)\rangle_3$, $j = 0, 1, 2$, the fluctuations of $X_1(t)$ and $X_2(t)$ read

$$\begin{aligned} \langle\triangle X_1^2(t)\rangle_0 &= \tfrac{1}{4}\,\mathrm{e}^{\alpha(t)}\left(2yc_{3,0}^2c_{3,2}^2 + 1\right), \\ \langle\triangle X_2^2(t)\rangle_0 &= \tfrac{1}{4}[k^{-2}\beta^2(t)\,\mathrm{e}^{\alpha(t)} + \mathrm{e}^{-\alpha(t)}]\left(2yc_{3,0}^2c_{3,2}^2 + 1\right), \end{aligned} \tag{33}$$

for the state $|\psi_{3,0}(z,t)\rangle$;

$$\begin{aligned} \langle\triangle X_1^2(t)\rangle_1 &= \tfrac{1}{4}\,\mathrm{e}^{\alpha(t)}\left(2yc_{3,1}^2c_{3,0}^2 + 1\right), \\ \langle\triangle X_2^2(t)\rangle_1 &= \tfrac{1}{4}[k^{-2}\beta^2(t)\,\mathrm{e}^{\alpha(t)} + \mathrm{e}^{-\alpha(t)}]\left(2yc_{3,1}^2c_{3,0}^2 + 1\right), \end{aligned} \tag{34}$$

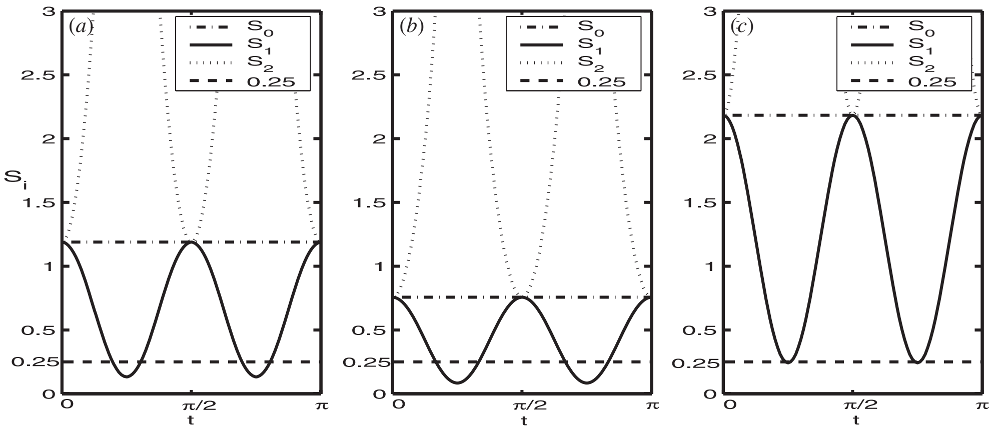

Figure 1. Dynamical quantum fluctuations $S_i(t) = \langle \triangle X_i^2(t)\rangle, i = 1, 2$, in three orthogonal normalized eigenstates $|\psi_j(z,t)\rangle_3$, $j = 0\,(a), 1\,(b), 2\,(c)$ of the operator $\hat{a}^3(t)$ for a TDHO with periodically varying frequency: $|z| = 0.8, \xi = 3, \eta = 0.8$. As a comparison, S_0 represents the quantum fluctuations in the three eigenstates of the operator $\hat{a}^3$ for a time-independent harmonic oscillator (i.e. $\eta = 0$).

for the state $|\psi_{3,1}(z,t)\rangle$ and

$$\begin{aligned} \left\langle \triangle X_1^2(t)\right\rangle_2 &= \tfrac{1}{4}\,\mathrm{e}^{\alpha(t)}\left(2yc_{3,2}^2c_{3,1}^2 + 1\right), \\ \left\langle \triangle X_2^2(t)\right\rangle_2 &= \tfrac{1}{4}[k^{-2}\beta^2(t)\,\mathrm{e}^{\alpha(t)} + \mathrm{e}^{-\alpha(t)}]\left(2yc_{3,2}^2c_{3,1}^2 + 1\right), \end{aligned} \tag{35}$$

for the state $|\psi_{3,2}(z,t)\rangle$, respectively.

In figure 1, we show the dynamical evolutions of the quantum fluctuations of two quadrature components of the phase amplitudes $X_1(t)$ and $X_2(t)$. It is seen that S_0 is always greater than 1/4 (see also in [6, 9]), which means that there is no squeezing effect in the eigenstates of the time-independent operator $\hat{a}^3$. However, it is shown that the component of S_1 could be less than 1/4 (see (*a*) and (*b*) in figure 1), which indicates that there exist dynamical squeezing effects in the two eigenstates $|\psi_j(z,t)\rangle$, $j = 0, 1$, of the time-dependent operator $\hat{a}^3(t)$.

5. Conclusions and discussions

The dynamical invariant $\hat{I}(t)$ for a general TDHO has been constructed by making use of two arbitrary linearly independent solutions to its classical equation of motion. With the complete set spanned by the instantaneous eigenstates of the constructed $\hat{I}(t)$, the exact solution to the time-dependent Schrödinger equation for such a TDHO could be obtained. Corresponding to such a dynamical invariant, a new pair of the time-dependent annihilation and creation operators ($\hat{a}(t)$ and $\hat{a}^\dagger(t)$) is introduced. Given the eigenstates of the operator $\hat{a}^N$ for the time-independent harmonic oscillator had attracted much attention, in this paper we constructed the exact eigenstates of the operator $\hat{a}^N(t)$ ($N \geqslant 3$) for the TDHO, and then investigated their mathematical and quantum statistical properties.

Specifically, a particle moving in a Paul trap with periodically varying frequency was treated as an example to demonstrate the dynamical squeezing effects for the eigenstates of the operator $\hat{a}^3(t)$. It is shown that the quantum statistical properties of these eigenstates for the

TDHO are different from those of the eigenstates of the operator $\hat{a}^3$ for the time-independent harmonic oscillator.

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